The non-bipartite graphs with all but two eigenvalues in $[-1, 1]$

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Abstract. The eigenvalues of a graph are those of its adjacency matrix. Recently, Cioabă, Haemers, Vermette, and Wong characterized all connected non-bipartite graphs with all but two eigenvalues equal to 1 or $-1$. In this article, we will generalize their result by explicitly determining all connected non-bipartite graphs with all but two eigenvalues in the interval $[-1, 1]$.

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1. Introduction

Throughout this article, all graphs are assumed to be simple, finite, and undirected. Let $G$ be a graph with the vertex set $\mathcal{V}(G)$ and the edge set $\mathcal{E}(G)$. The adjacency matrix of $G$, denoted by $\mathcal{A}(G)$, is a matrix whose entries are indexed by $\mathcal{V}(G) \times \mathcal{V}(G)$ and the $(u, v)$-entry is 1 if $\{u, v\} \in \mathcal{E}(G)$ and 0 otherwise. We use the notation $\phi_M(x)$ for the characteristic polynomial of a matrix $M$. The characteristic polynomial of $G$ is defined as $\phi_{\mathcal{A}(G)}(x)$ and is denoted by $\phi_G(x)$. 

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The zeros of \( \varphi_G(x) \) are said to be the eigenvalues of \( G \). Since \( \mathcal{A}(G) \) is a real symmetric matrix, all eigenvalues of \( G \) are real. We arrange the eigenvalues of \( G \) in non-increasing order and denote them by \( \lambda_1(G) \geq \cdots \geq \lambda_n(G) \), where \( n = |V(G)| \).

There are a lot of articles in the literature on graph spectra examining the graphs in which one or more of their eigenvalues are contained in a certain subset of real numbers. For instance, Petrović in [?] completely characterized the class of graphs \( G \) satisfying \( 0 < \lambda_2(G) < \sqrt{2} - 1 \), and, in addition, Torgašev in [?] completely characterized the class of graphs \( G \) on \( n \) vertices satisfying \(-1 < \lambda_{n-1}(G) < 0 \). It is noteworthy that, for any graph \( G \) on \( n \) vertices without isolated vertices, we know from [?] that \( \lambda_2(G) \leq 0 \) if and only if \( G \) is a complete multipartite graph, and also, by a simple argument given in Page 148 of [?], we know that \( \lambda_{n-1}(G) \geq 0 \) if and only if \( G \) is a complete bipartite graph. In this context, we address a situation in which both of the second smallest and the second largest eigenvalues are simultaneously involved. In [?], van Dam and Spence described the connected bipartite graphs with all but two eigenvalues in \([-1, 1]\). Very recently, Cioabă, Haemers, Vermette, and Wong in [?] explicitly determined the connected non-bipartite graphs with all but two eigenvalues in \([-1, 1]\). Petrović in [?] characterized the connected bipartite graphs with all but two eigenvalues in \([-1, 1]\), and before that, the trees with all but two eigenvalues in \([-1, 1]\) have been explicitly determined by Neumaier in [?]. In the present article, we characterize the connected non-bipartite graphs with all but two eigenvalues in \([-1, 1]\). Our characterization is quite explicit and the proof of our main theorem is based on the forbidden subgraph method.

Denote the complete graph, the path graph, and the cycle graph on \( n \) vertices by \( K_n \), \( P_n \), and \( C_n \), respectively. By applying Cauchy’s interlacing theorem [?, Corollary 2.5.2], it is readily seen that the largest eigenvalue of a graph \( G \) is at most 1 if and only if \( G \) is a vertex disjoint union of some copies of \( K_1 \) and \( K_2 \). Also, the smallest eigenvalue of a graph \( G \) is at least \(-1 \) if and only if \( G \) is a vertex disjoint union of some complete graphs. To be more clear, let \( \mathcal{G} \) denote the family of connected non-bipartite graphs all of whose eigenvalues are contained \([-1, 1]\) except the largest and smallest ones. As mentioned before, in this article, we will completely describe the family \( \mathcal{G} \). This generalizes the main result of [?].

Before proceeding, let us set some more notation. For every vertex \( v \) of a graph \( G \) and subset \( X \) of \( V(G) \), we let \( N_X(v) = \{ x \in X \mid \{ v, x \} \in \mathcal{E}(G) \} \) and we denote the induced subgraph of \( G \) on \( X \) by \( (X) \). For convenience, we write \( N(v) \) and \( \langle v_1, \ldots, v_k \rangle \) instead of \( N_{V(G)}(v) \) and \( \langle \{ v_1, \ldots, v_k \} \rangle \), respectively. For every pair of disjoint subsets \( X \) and \( Y \) of \( V(G) \), we denote by \( \langle X, Y \rangle \) the subgraph of \( G \) with the vertex set \( X \cup Y \) and the edge set \( \{ \{ x, y \} \in \mathcal{E}(G) \mid x \in X \text{ and } y \in Y \} \). Further, for each eigenvalue \( \lambda \) of a matrix \( M \), the eigenspace of \( M \) corresponding to \( \lambda \) is denoted by \( \mathcal{E}_M(\lambda) \) and whose dimension is said to be the multiplicity of \( \lambda \). For simplicity, we use the notation \( \mathcal{E}_G(\lambda) \) instead of \( \mathcal{E}_{\mathcal{A}(G)}(\lambda) \). We mean by \( \mathbb{R}^{V(G)} \) the set of real vectors whose components are indexed by \( V(G) \). The restriction of a vector \( x \in \mathbb{R}^{V(G)} \) to a subset \( X \subseteq V(G) \) is denoted by \( x|_X \).

2. Some subfamilies of \( \mathcal{G} \)
Let $\widehat{G}$ denote the family of graphs whose second largest and second smallest eigenvalues are contained in $[-1, 1]$. In this section, we introduce some subfamilies of $\widehat{G}$ that will be used in the next section. In the remainder of the article, we frequently apply the following theorem which is a consequence of Cauchy’s interlacing theorem [7, Corollary 2.5.2].

**Theorem 1.** Let $G$ be a graph on $n$ vertices and let $v \in V(G)$. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_{n-1}$ are respectively the eigenvalues of $G$ and $G - v$, then $\lambda_{i+1} \leq \lambda'_i \leq \lambda_i$ for any $i \in \{1, \ldots, n-1\}$.

We recall here the equitable partitions of matrices and then state a related well known result. Consider a symmetric matrix $M$ whose rows and columns are indexed by a set $\mathcal{P}$ and consider a partition $\Pi = \{P_1, \ldots, P_m\}$ of $\mathcal{P}$. The partition of $M$ with respect to $\Pi$ is said to be equitable if each submatrix $M_{i,j}$ formed by the rows whose indices are in $P_i$ and the columns whose indices are in $P_j$ has constant row sums, say $q_{i,j}$. The matrix $Q = [q_{i,j}]_{i,j \in \mathcal{P}}$ is called the quotient matrix of $M$ with respect to $\Pi$. The following fact is easy to prove and can be found in Page 24 of [7].

**Lemma 2.** With the above notation, any eigenvalue of $Q$ is also an eigenvalue of $M$.

Let us describe the figures in this section. Consider a graph $G$ which we draw in a figure and let $X, Y$ be two disjoint subsets of $V(G)$. For every $x \in X$ and $y \in Y$, if the points corresponding to $x$ and $y$ are joined by a full thin line, then $(x, y) \in E(G)$. Two ellipses corresponding to $X$ and $Y$ are joined with a full bold line if each pair in $X \times Y$ forms an edge of $G$ except those pairs whose corresponding points are joined by a dotted thin line. A solid black ellipse indicates a clique of $G$. We denote an ellipse containing $k$ vertices of $G$ with $S_k$ and we employ the notation $S'_k$ if there are two distinct ellipses of the size $k$ in the figure.

In worth to mention that, for any given graph $G$, a vector $x \in \mathbb{R}^{V(G)}$ is contained in $E_G(\lambda)$ if and only if

$$\lambda x_v = \sum_{w \in N(v)} x_w, \quad (1)$$

for all $v \in V(G)$.

**Lemma 3.** The family $\mathcal{G}_1$ depicted in Figure ?? is a subfamily of $\widehat{G}$.

**Proof.** Fix $G \in \mathcal{G}_1$ and let $n = |V(G)|$. We may assume that none of $a, a', b, b', c, d, d'$ is zero by noting Theorem ???. Let

$$W_0 = \left\{ x \in \mathbb{R}^{V(G)} \left| \sum_{w \in S_a} x_w = \sum_{w \in S_{a'}} x_w = 0 \text{ and } x_{\left| V(G) \setminus (S_a \cup S_{a'}) \right|} = 0 \right. \right\},$$

$$V_1 = \left\{ x \in \mathbb{R}^{V(G)} \left| \sum_{w \in S_b \cup S_c \cup S_d} x_w = \sum_{w \in S_{b'} \cup S_{c'} \cup S_{d'}} x_w = 0 \text{ and } x_{\left| S_a \cup S_{a'} \right|} = 0 \right. \right\}.$$
Figure 1. The family $\mathcal{G}_1$.

and

\[ V_{-1} = \left\{ x \in \mathbb{R}^{V(G)} \mid \sum_{w \in S_a \cup S_c} x_w = 0 \quad \text{and} \quad x_{v^{-1}} = 0 \quad \text{for} \quad \sum_{w \in S_{v^{-1}} \cup S_{v^{-1} \prime}} x_w = 0 \right\}. \]

For $\varepsilon \in \{-1, 1\}$, assume that $W_\varepsilon$ is the subset of $V_e$ consisting of any vector $x$ with $x_u = \varepsilon x_v$ for each full thin line $u \rightarrow v$ and $x_u = -\varepsilon x_v$ for each dotted thin line $u \rightarrow v$ in Figure 1. In view of (1), it is easily checked that $W_\varepsilon \subseteq \mathcal{G}_1(\varepsilon)$, for any $\varepsilon \in \{-1, 0, 1\}$. Since $\dim W_0 = a + a' - 2$ and $\dim W_1 = \dim W_{-1} = b + b' + c + d + d' - 2$, at least $n-6$ eigenvalues of $G$ are contained in $\{-1, 0, 1\}$.

For the remaining six eigenvalues of $G$, in view of Lemma 1, we consider the quotient matrix of $\mathcal{A}(G)$ corresponding to the partition $\{S_a, S_b, S_{b'}, S_c, S_{2d}, S_{a'}, S_{b'}, S'C, S_{2d'}\}$, say $Q$. From Theorem 1, one may assume that $a = a'$, $b = b'$, and $d = d'$. We have

\[ Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_1 \end{bmatrix}, \]

where

\[ Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} a & b & 0 & c & 2d \\ a & b & 0 & c & 2d \\ a & b & 0 & c & 2d \\ a & b & 0 & c & 2d \\ a & b & 0 & c & 2d \end{bmatrix}. \]
With an easy calculation, we find that $\varphi_G(x) = (x - 1)^3(x + 1)\psi_1(x)\psi_2(x)$, where

$$\psi_1(x) = x^3 - (a + b + c + 2d)x^2 + (c - 2d - 1)x + a$$

and

$$\psi_2(x) = x^3 + (a + b + c + 2d)x^2 + (c + 2d - 1)x - a.$$ 

Since

$$\begin{cases}
\psi_1(1) = -(b + 4d) < 0; \\
\psi_1(0) = a > 0; \\
\psi_1(-1) = -(b + 2c) < 0
\end{cases}$$

and

$$\begin{cases}
\psi_2(1) = b + 2c + 4d > 0; \\
\psi_2(0) = -a < 0; \\
\psi_2(-1) = b > 0,
\end{cases}$$

the intermediate value theorem yields that each of the polynomials $\psi_1$ and $\psi_2$ has exactly two roots in $(-1, 1)$. This shows that $\varphi_G$ has exactly one root in each of the intervals $(1, +\infty)$ and $(-\infty, -1)$, as required.

**Lemma 4.** The family $\mathcal{G}_2$ depicted in Figure ?? is a subfamily of $\hat{G}$.

![Figure 2. The family $\mathcal{G}_2$.](image)

**Proof.** Let $G \in \mathcal{G}_2$ and $n = |V(G)|$. By Theorem ??, we may assume that none of $a, b, c$ is zero. Define

$$W_0 = \left\{ x \in \mathbb{R}^{|V(G)|} \mid \sum_{w \in S_a} x_w = 0 \text{ and } x_{|V(G)\backslash S_a} = 0 \right\},$$

$$V_1 = \left\{ x \in \mathbb{R}^{|V(G)|} \mid \sum_{w \in S_b} x_w + 2 \sum_{w \in S_c} x_w = \sum_{w \in S_3} x_w = 0 \text{ and } x_{|S_b \cup S_c} = 0 \right\}.$$
\[
V_{-1} = \left\{ x \in \mathbb{R}^{|V(G)|} \middle| \sum_{w \in S_b} x_w = \sum_{w \in S_3} x_w = 0 \text{ and } x|_{S_j \cup \bar{S}_w} = 0 \right\}.
\]

For \(\varepsilon \in \{-1, 1\}\), assume that \(W_\varepsilon\) is the subset of \(V_{\varepsilon}\) consisting of any vector \(x\) with \(x_u = \varepsilon x_v\) for each full thin line \(u \rightarrow v\) and \(x_u = -\varepsilon x_v\) for each dotted thin line \(u \rightarrow v\) in Figure ???. In view of (??), it is easily verified that \(W_\varepsilon \subseteq E_G(\varepsilon)\), for any \(\varepsilon \in \{-1, 0, 1\}\). Since

\[
\dim W_0 = a - 1 \quad \text{and} \quad \dim W_1 = \dim W_{-1} = b + c + 1,
\]

at least \(n - 6\) eigenvalues of \(G\) are contained in \([-1, 0, 1]\). For the remaining six eigenvalues of \(G\), in view of Lemma ??, we consider the quotient matrix of \(\mathcal{A}(G)\) corresponding to the partition \(\Pi = \{S_3, S'_3, S_1, S_a, S_b, S'_b, S_2\}\), say \(Q\). We have

\[
Q = \begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & a & b & 0 & 2c \\
0 & 0 & 0 & a & b & 0 & 2c \\
0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

(3)

With an easy calculation, we find that \(\varphi(Q) = (x + 1)(x - 1)^2\psi(x)\), where

\[
\psi(x) = x^4 - 4(a + b + 2c + 1)x^2 - 8cx + 4a.
\]

It follows from

\[
\begin{cases}
\psi(1) = -(4b + 16c + 3) < 0;
\psi(0) = 4a > 0;
\psi(-1) = -(4b + 3) < 0
\end{cases}
\]

that \(\psi(x)\) has zero in each of the intervals \((-1, 0)\) and \((0, 1)\). Since \(\psi(x)\) is of degree 4, \(\psi(x)\) has exactly two roots outside \([-1, 1]\). To finish the proof, we show that the eigenvalue spectrum of \(G\) is equal to the multiset

\[
\left\{ (-1)[b + c + 2], \ [a - 1], \ [b + c + 2] \right\} \cup \left\{ x \in \mathbb{R} | \psi(x) = 0 \right\},
\]

where the exponents indicate the multiplicities of the eigenvalues. To do this, letting \(\varepsilon \in \{-1, 1\}\), it is enough to demonstrate that \(E_G(\varepsilon) \neq W_\varepsilon\) by noting (??). Consider the eigenvectors

\[
y_1 = \left( -2, -1, 3, 0, \frac{1}{b}, \frac{1}{b}, \frac{1}{c} \right) \in \mathcal{E}_Q(1) \quad \text{and} \quad y_{-1} = \left( -2, 1, -3, 0, \frac{3}{b}, \frac{3}{b}, -\frac{1}{c} \right) \in \mathcal{E}_Q(-1).
\]

Now, if \(M\) is the matrix whose columns are the characteristic vectors of the elements of \(\Pi\), then \(My_\varepsilon \in E_G(\varepsilon) \setminus W_\varepsilon\), completing the proof.

\[
\square
\]

**Lemma 5.** The family \(G_3\) depicted in Figure ?? is a subfamily of \(\hat{G}\).
Proof. Fix $G \in \mathcal{G}_3$ and let $n = |V(G)|$. From Theorem ??, we may assume that none of $a, b, c$ is zero. Define

$$W_0 = \left\{ x \in \mathbb{R}^{V(G)} \middle| \sum_{w \in S_a} x_w = 0 \text{ and } x_{|V(G)\backslash S_a} = 0 \right\},$$

and by assuming $S_1 = \{t\}$, let

$$V_1 = \left\{ x \in \mathbb{R}^{V(G)} \middle| \sum_{w \in S_b} x_w + 2 \sum_{w \in S_c} x_w = -\frac{1}{2} \sum_{w \in S_3} x_w = -x_t \text{ and } x_{|S_a} = 0 \right\},$$

and

$$V_{-1} = \left\{ x \in \mathbb{R}^{V(G)} \middle| \sum_{w \in S_b} x_w = -\frac{1}{2} \sum_{w \in S_3} x_w = x_t \text{ and } x_{|S_a} = 0 \right\}.$$

For $\varepsilon \in \{-1, 1\}$, assume that $W_\varepsilon$ is the subset of $V_\varepsilon$ consisting of any vector $x$ with $x_u = \varepsilon x_v - x_t$, for each full thin line $u \rightarrow v$ with $u \in S_3'$, and $x_u = \varepsilon x_v$ for each other full thin line $u \rightarrow v$ in Figure ???. By (??), it is easily checked that $W_\varepsilon \subseteq \delta_G(\varepsilon)$, for any $\varepsilon \in \{-1, 0, 1\}$. Since $\dim W_0 = a - 1$ and $\dim W_1 = \dim W_{-1} = b + c + 2$, at least $n - 4$ eigenvalues of $G$ are contained in $\{-1, 0, 1\}$. For the remaining four eigenvalues of $G$, using Lemma ??, we consider the quotient matrix of $\mathcal{A}(G)$ corresponding to the partition $\{S_3', S_3, S_1, S_a, S_b, S_b', S_{2c}\}$, say $Q$. 

Figure 3. The family $\mathcal{G}_3$. 

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We have
\[
Q = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & a & b & 0 & 2c \\
3 & 0 & 0 & a & b & 0 & 2c \\
0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

An easy computation shows that \(\varphi_Q(x)\) is equal to the characteristic polynomial of the matrix given in (??) and so \(\varphi_Q(x)\) has exactly one root in each of the intervals \((-\infty, -1), (-1, 0), (0, 1),\) and \((1, +\infty)\). The proof is complete.

\[\square\]

Remark 6. The proofs of Lemmas ?? and ?? show that if the parameters \(a, b, c\) of the graphs \(G \in G_2\) and \(G' \in G_3\) are all non-zero and correspondingly equal, then \(G\) and \(G'\) are two non-isomorphic graphs with the same eigenvalue spectra.

For every \(r_1, \ldots, r_n \in \mathbb{R}\), we define \(\zeta(r_1, \ldots, r_n)\) to be the multiplicity of 0 in the multiset \(\{r_1, \ldots, r_n\}\).

Lemma 7. Let \(a \geq 3\) and \(b \geq 1\). The family \(G_4\) depicted in Figure 4 is a subfamily of \(\mathcal{G}\) if and only if \((a-2)(b+c+d-3) \leq 2\).

\[\text{Figure 4. The family } G_4.\]

Proof. Let \(G \in G_4\) and \(n = |V(G)|\). In view of (??), it is easily checked that
\[
\left\{ x \in \mathbb{R}^{V(G)} \left| \sum_{w \in S_c} x_w = \sum_{w \in S_d} x_w = 0 \text{ and } x_{|S_a \cup S_b \cup S'_b} = 0 \right. \right\} \subseteq \mathcal{E}_G(0).
\]

Let
\[
W_1 = \left\{ x \in \mathbb{R}^{V(G)} \left| \sum_{w \in S_b} x_w = 0 \text{ and } x_{|S_a \cup S_c \cup S_d} = 0 \right. \right\}
\]
and

\[ W_{-1} = \left\{ x \in \mathbb{R}^{\mathcal{V}(G)} \left| \sum_{w \in S_a} x_w = \sum_{w \in S_b} x_w = 0 \text{ and } x|_{S_c \cup S_d} = 0 \right. \right\}. \]

If \( \varepsilon \in \{-1, 1\} \), then any vector \( x \in W_\varepsilon \) with \( x_u = -\varepsilon x_v \), for each dotted thin line \( u \cdots v \) in Figure ??, is contained in \( \mathcal{E}_G(\varepsilon) \). Thus, at least \( n + \zeta(c, d) - 5 \) eigenvalues of \( G \) are contained in \( \{-1, 0, 1\} \). Let

\[
Q = \begin{bmatrix}
  a - 1 & 0 & 0 & b & d \\
  0 & 0 & 0 & b - 1 & d \\
  0 & 0 & 0 & b & d \\
  a & b - 1 & c & 0 & 0 \\
  a & b & c & 0 & 0
\end{bmatrix}.
\]

It is not hard to see that the characteristic polynomial of the quotient matrix of \( \mathcal{A}(G) \) corresponding to the partition of \( \mathcal{V}(G) \) obtained from the non-empty elements of the ordered multiset \( \{S_a, S_b, S_c, S_d\} \) is equal to \( \varphi_Q(x)/x^{\zeta(c, d)} \). Thus, to finish the proof, it is enough, in view of Lemma ??, to find the necessary and sufficient conditions for which \( \varphi_Q(x) \) has at most one root in each of the intervals \((-\infty, -1)\) and \((1, +\infty)\). With an easy calculation, we obtain that

\[
\varphi_Q(x) = x^5 - (a - 1)x^4 - ((a + b + c)(b + d) - 2b + 1)x^3 + (a - 1)((b + c)(b + d) - 2b + 1)x^2 + d(a + c)x - cd(a - 1)
\]

and so

\[
\begin{align*}
\varphi_Q(1) &= b((a - 2)(b + c + d - 3) - 2); \\
\varphi_Q(0) &= -cd(a - 1) \leq 0; \\
\varphi_Q(-1) &= ab(b + c + d - 1) \geq 0.
\end{align*}
\]

We recall that the largest eigenvalue of \( G \) is greater than 1. Furthermore, the Perron–Frobenius theorem [??, Theorem 2.2.1] implies that the largest root of \( \varphi_G(x) \) is simple. So, \( \varphi_Q(x) \) has at least two zeros in \((1, +\infty)\) provided \( \varphi_Q(1) > 0 \). Therefore, we may assume that \( \varphi_G(1) \leq 0 \). Since \( \varphi_G \) has at least one root in \([-1, 0]\), in order to prove that \( \varphi_G(x) \) has at most one zero in \((-\infty, -1)\) and exactly one zero in \((1, +\infty)\), it suffices to show that \( \varphi_G(r) > 0 \) for some \( r \in (0, 1) \). It follows from \( \varphi_G(1) \leq 0 \) that \((a - 2)(b + c + d - 3) \leq 2 \). Hence, we can distinguish the following cases.

**Case 1.** \( b + c + d \leq 3 \). From Theorem ??, it is enough to consider only the case \( b + c + d = 3 \). By considering all possible values for \( b, c, d \) with \( b \geq 1 \) and after some computations, one can conclude using \( a \geq 3 \) that \( \varphi_G \left( \frac{11}{2b} \right) \geq 0 \), as desired.

**Case 2.** \( b + c + d \geq 4 \). Using \( \varphi_G(1) \leq 0 \) and Theorem ??, we may assume that \( a \in \{3, 4\} \) and \( a + b + c + d = 8 \). By considering all possible values for \( a, b, c, d \) with \( b \geq 1 \), it is not hard to demonstrate that \( \varphi_G \left( \frac{1}{3} \right) \geq 0 \), as required.

Now, the proof of the lemma is complete.

\[ \square \]

**Lemma 8.** Let \( a \geq 3 \). The family \( \mathcal{G}_5 \) depicted in Figure ?? is a subfamily of \( \hat{\mathcal{G}} \) if and only if one of the following holds.

\[ 9 \]
(i) \( ab \leq a + 2b \) and \( (d,e) \neq (0,0) \).

(ii) \( (a - 2)(bc - 1) \leq ac \) and \( d = e = 0 \).

Figure 5. The family \( \mathcal{G}_5 \).

**Proof.** Let \( G \in \mathcal{G}_5 \) and \( n = |\mathcal{V}(G)| \). Note that the conditions \( ab \leq a + 2b \) and \( (a - 2)(bc - 1) \leq ac \) are trivially satisfied if \( b = 0 \). Thus, in view of Theorem ??, we may assume that \( b \geq 1 \). From (??), it is easily verified that

\[
\left\{ x \in \mathbb{R}^{\mathcal{V}(G)} \left| \sum_{w \in S_b} x_w = \sum_{w \in S_c} x_w = 0 \text{ and } x_{|\mathcal{V}(G)\setminus(S_b\cup S_c)} = 0 \right. \right\} \subseteq \mathcal{E}_G(0).
\]

Let

\[
W_1 = \left\{ x \in \mathbb{R}^{\mathcal{V}(G)} \left| \sum_{w \in S_d} x_w + 2 \sum_{w \in S_e} x_w = 0 \text{ and } x_{|S_d\cup S_e} = 0 \right. \right\}
\]

and

\[
W_{-1} = \left\{ x \in \mathbb{R}^{\mathcal{V}(G)} \left| \sum_{w \in S_a} x_w = \sum_{w \in S_d} x_w = \sum_{w \in S_e} x_w = 0 \text{ and } x_{|S_a\cup S_e} = 0 \right. \right\}.
\]

If \( \varepsilon \in \{-1, 1\} \), then any vector \( x \in W_\varepsilon \) with \( x_u = \varepsilon x_v \), for each full line \( u \to v \) in Figure ??, is contained in \( \mathcal{E}_G(\varepsilon) \). Therefore, at least \( n + \zeta(c, d, e, d + e) - 6 \) eigenvalues of \( G \) are contained in
\{−1,0,1\}. Let

\[ Q = \begin{bmatrix}
  a-1 & 0 & c & d & 0 & 2e \\
  0 & 0 & c & d & 0 & 2e \\
  a & b & 0 & 0 & 0 & 0 \\
  a & b & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  a & b & 0 & 0 & 0 & 1 \\
\end{bmatrix}. \]

It is not hard to verify that the characteristic polynomial of the quotient matrix of \( A(G) \) corresponding to the partition of \( V(G) \) obtained from the non-empty elements of the ordered multiset \( \{S_a, S_b, S_c, S_d, S'_a, S_e\} \) is equal to

\[ f(x) = \frac{\varphi_Q(x)}{x^{c(e)}(x-1)^{c(d,e)}(x+1)^{c(d)}}. \] (4)

With an easy calculation, we find that \( \varphi_Q(x) = (x-1)\psi(x) \), where

\[ \psi(x) = x^5 - (a - 1)x^4 - \left( (a + b)(c + d + 2e) + 1 \right)x^3 \\
+ \left( (a - 1)(bc + bd + 2be + 1) - 2e(a + b) \right)x^2 + \left( c(a + b) + 2be(a - 1) \right)x - bc(a - 1). \]

We discern the following cases.

**Case 1.** Assume that \((d, e) \neq (0, 0)\).

From (??) and using Lemma ??, it is enough to obtain the necessary and sufficient conditions for which \( \varphi_Q(x) \) has at most one root in each of the intervals \((-\infty, -1)\) and \((1, +\infty)\). We have

\[ \psi(1) = (ab - a - 2b)(d + 4e). \] (5)

Further, we know from the Perron–Frobenius theorem [??, Theorem 2.2.1] that the largest root of \( \psi(x) \) is simple. Thus, it follows from (??) and the assumption \((d, e) \neq (0, 0)\) that \( \psi(x) \) has at most one zero in \((1, +\infty)\) if \( ab \leq a + 2b \). So, we may assume that \( ab \leq a + 2b \). We want to show that \( \psi(x) \) has three roots in \([-1, 1]\). Theorem ?? allows us to assume that \( 2e \geq c \). Since

\begin{align*}
\psi\left(\frac{1}{2}\right) &= \frac{1}{16}((2a - 3)(2b - 1) - 3)(-3c + d + 6e) + \frac{3}{32}(2a - 3) > 0; \\
\psi(0) &= -bc(a - 1) \leq 0; \\
\psi(-1) &= ad(b + 1) \geq 0,
\end{align*}

we are done by the intermediate value theorem.

**Case 2.** Assume that \( d = e = 0 \).

The structure of \( G \), depicted in Figure ??, makes it possible to assume that \( c \geq 1 \). It follows from (??) that \( f(x) = x^3 - (a - 1)x^2 - c(a + b)x + bc(a - 1) \). We have

\begin{align*}
f(1) &= (a - 2)(bc - 1) - ac; \\
f(0) &= bc(a - 1) > 0.
\end{align*}
Applying the Perron–Frobenius theorem [?, Theorem 2.2.1], we deduce that \( f(x) \) has a simple zero in \((1, +\infty)\) and hence the necessary and sufficient conditions for which \( f(x) \) has exactly one zero in each of the intervals \((-\infty, -1)\) and \((1, +\infty)\) will be obtained from \( f(1) \leq 0 \). This completes the proof of the lemma.

\[\square\]

**Lemma 9.** Let \( a, c \geq 3 \). The family \( \mathcal{G}_6 \) depicted in Figure ?? is a subfamily of \( \hat{\mathcal{G}} \) if and only if \( b, d \in \{0, 1, 2\} \) and \((ab - a - 2b)(cd - c - 2d) \geq (a - 2)(c - 2)\).

![Figure 6. The family \( \mathcal{G}_6 \).](image)

**Proof.** Let \( G \in \mathcal{G}_6 \) and \( n = |\mathcal{V}(G)| \). The structure of \( G \) which is depicted in Figure ?? allows us to assume that \((b, d) \neq (0, 0)\). By (??), it is straightforwardly checked that

\[
\left\{ x \in \mathbb{R}^{\mathcal{V}(G)} : \sum_{w \in S_b} x_w = \sum_{w \in S_d} x_w = 0 \text{ and } x|_{S_b \cup S_c} = 0 \right\} \subseteq \mathcal{E}(0)
\]

and

\[
\left\{ x \in \mathbb{R}^{\mathcal{V}(G)} : \sum_{w \in S_a} x_w = \sum_{w \in S_c} x_w = 0 \text{ and } x|_{S_b \cup S_d} = 0 \right\} \subseteq \mathcal{E}(-1).
\]

So, at least \( n + \zeta(b, d) - 4 \) eigenvalues of \( G \) are contained in \([-1, 0]\). Let

\[
Q = \begin{bmatrix}
a -1 & 0 & c & d \\
0 & 0 & c & d \\
a & b & c - 1 & 0 \\
a & b & 0 & 0
\end{bmatrix}.
\]

It is not hard to observe that the characteristic polynomial of the quotient matrix of \( \mathcal{A}(G) \) corresponding to the partition of \( \mathcal{V}(G) \) obtained from the non-empty elements of the ordered multiset \( \{S_a, S_b, S_c, S_d\} \) is equal to \( \varphi_G(x)/x^{\zeta(b, d)} \). Hence, to finish the proof, it is enough, by Lemma ??,
to find the necessary and sufficient conditions for which \( \varphi_Q(x) \) has at most one root in each of the intervals \((-\infty, -1)\) and \((1, +\infty)\). With an easy calculation, we obtain that

\[
\varphi_Q(x) = x^4 - (a + c - 2)x^3 - (ad + bc + bd + a + c - 1)x^2
+ (abc + abd + acd + bcd - ad - bc - 2bd)x - bd(a - 1)(c - 1)
\]

and so

\[
\begin{align*}
\varphi_Q(1) &= (a - 2)(c - 2) - (ab - a - 2b)(cd - c - 2d); \\
\varphi_Q(0) &= -bd(a - 1)(c - 1) \leq 0; \\
\varphi_Q(-1) &= -ac(bd + b + d) \leq 0.
\end{align*}
\]

Since \(a, c \geq 3\) and the second largest eigenvalue of the graph depicted in Figure ?? is greater than 1, we conclude that \(b, d \in \{0, 1, 2\}\). Further, we deduce from the Perron–Frobenius theorem [?, Figure 7.]

Theorem 2.2.1] that \( \varphi_Q(x) \) has a simple zero in \((1, +\infty)\). If \( \varphi_Q(1) > 0 \), then \( \varphi_Q(x) \) must have more than one root in \((1, +\infty)\). Thus, we may assume that \( \varphi_Q(1) \leq 0 \). Obviously, if we show that \( \varphi_Q(r) \geq 0 \) for some \( r \in [0, 1) \), then \( \varphi_Q(x) \) must have exactly one root in each of the intervals \((-\infty, -1)\) and \((1, +\infty)\). Therefore, the existence of such an \( r \) will complete the proof.

By symmetry, it suffices to assume that \( b \geq d \). We have

\[
\begin{align*}
\varphi_Q(0) &= 0, & \text{if } d = 0; \\
\varphi_Q \left( \frac{1}{2} \right) &= \frac{3}{8}(a + c - 6) + \frac{9}{16}, & \text{if } b = d = 1; \\
\varphi_Q \left( \frac{3}{2} \right) &= \frac{40}{27}(a - 3) + \frac{10}{27}(c - 3) + \frac{100}{81}, & \text{if } b = 2 \text{ and } d = 1; \\
\varphi_Q \left( \frac{7 - \sqrt{33}}{2} \right) &= 0, & \text{if } b = d = 2.
\end{align*}
\]

Notice that, in the case \( b = d = 2 \), it follows from \( \varphi_Q(1) \leq 0 \) that \( a = c = 3 \). The proof is now complete.

\[
\square
\]

3. The main theorem

In the previous section, we showed that \( \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_6 \subseteq \widehat{\mathcal{G}} \). In this section, we prove that \( \mathcal{G} \subseteq \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_6 \). We start with a list of forbidden induced subgraphs.
Lemma 10. No graph in $\hat{G}$ has one of the graphs presented in Figures ??–?? as an induced subgraph.

Proof. We employ the computer software newGRAPH to verify that each graph in Figures ??–?? has either the second largest eigenvalue strictly greater than 1 or the second smallest eigenvalue strictly less than $-1$. So, the result follows from Theorem ??.

![Figure 8. Some forbidden induced subgraphs on 5 vertices.](image)

![Figure 9. Some forbidden induced subgraphs on 6 vertices.](image)

Theorem 11. Let $G \in \hat{G}$. Then $G$ has one of the forms presented in Figures ??–??.

Proof. We will frequently use Lemma ?? to determine the local structure of $G$. Since $G$ has no induced subgraph isomorphic to $F_1$ or $F_7$, it follows that $G$ contains no induced odd cycle of
the length five or more. Also, since $G$ is not bipartite, the clique number of $G$ is at least 3. For each $X \subseteq \mathcal{V}(G)$, define $\partial X$ as the set of all outgoing edges of $X$. Fix a maximum clique $\Omega$ of $G$ such that

$$|\partial \Omega| = \min \{ |\partial X| \mid X \text{ is a maximum clique of } G \}.$$  

(6)

As the first step, we establish some facts about $G$. The first one, which has a key role in our proof, is established in Lemma 4 of [?]. We include its proof only for the convenience of the reader.

**Fact 1.** There exists a partition $\Omega = P \cup Q$ such that for each $v \in \mathcal{V}(G) \setminus \Omega$, either $N_{\Omega}(v) = \emptyset$, $N_{\Omega}(v) = P$, or $N_{\Omega}(v) = Q$.

We may assume that there exist the distinct vertices $a, b \in \mathcal{V}(G) \setminus \Omega$ with $N_{\Omega}(a) \neq \emptyset$ and $N_{\Omega}(b) \neq \emptyset$. First, assume that $N_{\Omega}(a) \cap N_{\Omega}(b) \neq \emptyset$. We will show that $N_{\Omega}(a) = N_{\Omega}(b)$. By contradiction, suppose that there exists $x \in N_{\Omega}(a) \setminus N_{\Omega}(b)$. Let $u \in N_{\Omega}(a) \cap N_{\Omega}(b)$. Since $\Omega$ is a maximum clique, there exists $v \in \Omega \setminus N(a)$. Now, since $\langle a, b, u, v, x \rangle$ is isomorphic to one of $F_4$, $F_5$, or $F_6$, we get a contradiction by noting Lemma ???. Next, assume that $N_{\Omega}(a) \cap N_{\Omega}(b) = \emptyset$. We show that $\Omega = N_{\Omega}(a) \cup N_{\Omega}(b)$. By contradiction, suppose that there exists $x \in \Omega \setminus (N_{\Omega}(a) \cup N_{\Omega}(b))$. Let $u \in N_{\Omega}(a)$ and $v \in N_{\Omega}(b)$. Since $\langle a, b, u, v, x \rangle$ is isomorphic to $F_2$ or $F_3$, we get a contradiction. This completes the proof of Fact 1.

By Fact 1, we may consider $\mathcal{V}(G) \setminus \Omega$ as a union of the mutually disjoint subsets

$$A = \{ x \in \mathcal{V}(G) \setminus \Omega \mid N_{\Omega}(x) = P \}.$$  

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\[ B = \{ x \in \mathcal{V}(G) \setminus \mathcal{O} \mid \mathcal{N}_G(x) = Q \}, \]
\[ C = \{ x \in \mathcal{V}(G) \setminus \mathcal{O} \mid \mathcal{N}_G(x) = \emptyset \}. \]

In the rest of the proof, we let
\[ P = \{ u_1, \ldots, u_{|P|} \} \quad \text{and} \quad Q = \{ v_1, \ldots, v_{|Q|} \}. \]

**Fact 2.** Each vertex in \( C \) has neighbors in exactly one of \( A \) or \( B \).

The proof of Fact 2 is by contradiction. If there exist the vertices \( c \in C, a \in \mathcal{N}_A(c) \), and \( b \in \mathcal{N}_B(c) \), then \( \langle a, b, c, u_1, v_1 \rangle \) is isomorphic to \( F_1 \) or \( F_3 \), a contradiction. If there exists a vertex in \( C \) with no neighbor in \( A \cup B \), then the connectivity of \( G \) yields that there are two adjacent vertices \( c_1, c_2 \in C \) with \( \mathcal{N}_{A \cup B}(c_1) \neq \emptyset \) and \( \mathcal{N}_{A \cup B}(c_2) = \emptyset \). Letting \( x \in \mathcal{N}_{A \cup B}(c_1) \), \( y \in \mathcal{N}_A(x) \), \( z \in \mathcal{O} \setminus \mathcal{N}(x) \), and \( t \in \mathcal{O} \setminus \{ y, z \} \), it is seen that \( \langle c_1, c_2, x, y, z, t \rangle \) is isomorphic to \( F_9 \) or \( F_{14} \), a contradiction. This completes the proof of Fact 2.

By Fact 2, we may consider \( C \) as a union of the mutually disjoint subsets
\[ C_A = \{ x \in C \mid \mathcal{N}_A(x) \neq \emptyset \} \quad \text{and} \quad C_B = \{ x \in C \mid \mathcal{N}_B(x) \neq \emptyset \}. \]

**Fact 3.** The subgraph \( \langle C \rangle \) has no edge.

The proof of Fact 3 is by contradiction. Suppose that there are two adjacent vertices \( c_1, c_2 \in C \). Fact 2 guarantees the existence of the vertices \( x_1 \in \mathcal{N}_{A \cup B}(c_1) \) and \( x_2 \in \mathcal{N}_{A \cup B}(c_2) \). If \( x_1 = x_2 \), then \( \langle c_1, c_2, w, x_1, x_2 \rangle \), for some \( w \in \mathcal{N}_A(x_1) \), \( w_2 \in \mathcal{O} \setminus \mathcal{N}(x_1) \), and \( w_3 \in \mathcal{O} \), is isomorphic to \( F_{13} \) or \( F_{15} \), a contradiction. Hence, we may suppose that \( \mathcal{N}_{A \cup B}(c_1) \cap \mathcal{N}_{A \cup B}(c_2) = \emptyset \). If \( x_1 \) and \( x_2 \) are distinct and both belong to one of \( A \) or \( B \), then \( \langle c_1, c_2, x, y, z, t \rangle \) is isomorphic to \( F_1 \) or \( F_3 \) for some \( w \in \mathcal{N}_A(x_1) \cap \mathcal{N}_A(x_2) \), which is a contradiction. Thus, without loss of generality, we may assume that \( x_1 \in \mathcal{N}_A(c_1) \) and \( x_2 \in \mathcal{N}_B(c_2) \) and \( |P| \geq 2 \). Now, \( \langle c_1, c_2, u_1, u_2, x_1, x_2 \rangle \) is isomorphic to \( F_9 \) or \( F_{12} \), a contradiction. This contradiction proves Fact 3.

**Fact 4.** Let \( a \in A \) and \( b \in B \) such that \( |\mathcal{N}_{P \cup A \cup C}(a)| \geq 2 \) and \( |\mathcal{N}_{Q \cup B \cup C}(b)| \geq 2 \). If \( \langle a, b \rangle \notin \mathcal{E}(G) \), then \( |P| = |Q| = 2 \) and \( \mathcal{N}_{A \cup C}(a) = \mathcal{N}_{B \cup C}(b) = \emptyset \).

Assume that \( \langle a, b \rangle \notin \mathcal{E}(G) \) and let \( x \in \mathcal{N}_{P \cup A \cup C}(a) \) and \( y \in \mathcal{N}_{Q \cup B \cup C}(b) \). If \( \langle a, y \rangle \notin \mathcal{E}(G) \), then Facts 1 and 2 show that \( y \in B \) and so \( \langle a, b, u_1, v_1, y \rangle \) is isomorphic to \( F_3 \), a contradiction. Therefore, \( \langle a, y \rangle \notin \mathcal{E}(G) \), and similarly, \( \langle b, x \rangle \notin \mathcal{E}(G) \). If \( \langle x, y \rangle \notin \mathcal{E}(G) \), then \( \langle x, y \rangle \notin P \times Q \) and so \( \langle a, b, u_1, v_1, x, y \rangle \) is isomorphic to one of \( F_7, F_9, F_{13}, F_{14}, \) or \( F_{15} \), which is a contradiction by noting Lemma 23. Thus, \( \langle x, y \rangle \in \mathcal{E}(G) \) and hence either \( (x, y) \in A \times B \) or \( (x, y) \in P \times Q \). But, the first one does not occur, since in that case \( \langle a, u_1, v_1, x, y \rangle \) is isomorphic to \( F_3 \), a contradiction. This yields that \( (x, y) \in P \times Q \), meaning \( \mathcal{N}_{P \cup A \cup C}(a) \subseteq P \) and \( \mathcal{N}_{Q \cup B \cup C}(b) \subseteq Q \). To finish the proof of Fact 4, it is enough to show that \( |P| = |Q| = 2 \). To get a contradiction, suppose, without loss of generality, that \( |P| \geq 3 \). But, \( \langle a, b, u_1, u_2, u_3, v_1, v_2 \rangle \) is isomorphic to \( F_{23} \). This contradiction completes the proof of Fact 4.

**Fact 5.** Every vertex in \( A \) having a neighbor in \( A \cup C \) is adjacent to all but at most one vertices in \( B \) and every vertex in \( B \) having a neighbor in \( B \cup C \) is adjacent to all but at most one vertices in \( A \).
Working towards a contradiction, suppose that a vertex \( a \in A \) has a neighbor \( x \in A \cup C \) and is not adjacent to two distinct vertices \( b_1, b_2 \in B \). It follows from Fact 4 that \( \{b_1, b_2\} \notin \mathcal{E}(G) \). If \( b_1, b_2 \notin \mathcal{N}(x) \), then \( \langle a, b_1, b_2, u_1, v_1, x \rangle \) is isomorphic to \( F_8 \) or \( F_{10} \), a contradiction. If a vertex \( y \in \{b_1, b_2\} \) is adjacent to \( x \), \( x \in A \) and \( \langle a, u_1, v_1, x, y \rangle \) is isomorphic to \( F_5 \), which is again a contradiction. This proves the first statement of Fact 5 and the similar argument works for the second statement.

**Fact 6.** No vertex in \( A \) has neighbors in both \( A \) and \( C \) and no vertex in \( B \) has neighbors in both \( B \) and \( C \).

Towards a contradiction, suppose that a vertex \( a \in A \) has two neighbors \( x \in A \) and \( y \in C \). The maximality of \( \Omega \) forces that \( |Q| \geq 2 \). If \( \{x, y\} \notin \mathcal{E}(G) \), then \( \langle a, u_1, v_1, x, y \rangle \) is isomorphic to \( F_2 \) and if \( \{x, y\} \in \mathcal{E}(G) \), then \( \langle a, u_1, v_2, x, y \rangle \) is isomorphic to \( F_{15} \), a contradiction. This establishes the first statement of Fact 6 and the similar argument works for the second statement.

**Fact 7.** The both \( \langle A \rangle \) and \( \langle B \rangle \) are disjoint unions of some copies of \( K_1 \) and \( K_2 \).

Arguing toward a contradiction, suppose that two distinct vertices \( a, b \in A \) have a common neighbor \( c \in A \). So, the maximality of \( \Omega \) yields that \( |Q| \geq 2 \). If \( \{a, b\} \notin \mathcal{E}(G) \), then \( \langle a, b, c, u_1, v_1 \rangle \) is isomorphic to \( F_4 \), a contradiction. If \( \{a, b\} \in \mathcal{E}(G) \), then \( \langle a, b, c, u_1, v_2 \rangle \) is isomorphic to \( F_{17} \), a contradiction. This obviously proves the result for \( \langle A \rangle \). The similar argument can be used for \( \langle B \rangle \).

We are now ready to determine the structure of \( G \). Recall that, as we saw above, \( \mathcal{V}(G) \) can be considered to be equal to \( P \cup Q \cup A \cup B \cup C \) in which \( \langle P \cup Q \rangle \) is a complete graph, \( \langle A \rangle \) and \( \langle B \rangle \) are disjoint unions of some copies of \( K_1 \) and \( K_2 \), and \( \langle C \rangle \) is an edgeless graph. Furthermore, \( \langle P, A \rangle \) and \( \langle Q, B \rangle \) are complete bipartite graphs, and, in addition, \( \langle P, B \rangle \), \( \langle Q, A \rangle \), and \( \langle P \cup Q, C \rangle \) are edgeless graphs. So, it remains to determine the subgraphs \( \langle A, B \rangle \), \( \langle A, C \rangle \), and \( \langle B, C \rangle \). To do this, we consider the following cases.

**Case 1.** Assume that there exists \( c \in C \) of degree at least 2.

Without loss of generality, we may assume that \( c \in C_B \). If \( |P| \geq 2 \), then \( \langle b_1, b_2, c, u_1, u_2, v_1 \rangle \), for some \( b_1, b_2 \in \mathcal{N}_b(c) \), is isomorphic to \( F_{12} \) or \( F_{15} \), a contradiction. This means that \( |P| = 1 \) and so \( |Q| \geq 2 \). We are going to show that \( \mathcal{E}(\langle A, C \rangle) \) is a matching. If a vertex \( x \in A \) has two distinct neighbors \( c_1, c_2 \in C \), then \( \langle c_1, c_2, u_1, v_1, v_2, x \rangle \) is isomorphic to \( F_{10} \), a contradiction. If a vertex \( x \in A \) has two distinct neighbors \( a_1, a_2 \in A \), then \( \langle a_1, a_2, u_1, v_1, v_2, x \rangle \) is isomorphic to \( F_{12} \) or \( F_{15} \), a contradiction. So, \( \mathcal{E}(\langle A, C \rangle) \) is a matching. Further, since \( |P| = 1 \) and \( \Omega \) is a maximum clique, one concludes that \( \mathcal{E}(\langle B \rangle) = \emptyset \). We discern the following cases.

**Case 1.1.** Assume that \( |Q| = 2 \) and \( \mathcal{N}(c) = B \).

First, we claim that \( \langle A, B \rangle \) is the graph obtained from a complete bipartite graph by deleting the edges of a matching. To see this, note that if a vertex \( x \in A \) is not adjacent to two distinct vertices \( b_1, b_2 \in B \), then \( \mathcal{E}(\langle B \rangle) = \emptyset \) yields that \( \langle b_1, b_2, c, u_1, v_1, x \rangle \) is isomorphic to \( F_{11} \), a contradiction. This and Fact 5 together establish the claim. Next, we show that \( \mathcal{E}(\langle B, C \setminus \{c\} \rangle) \) is a matching. If a vertex \( x \in B \) has two distinct neighbors \( y_1, y_2 \in C \setminus \{c\} \), then, using Fact 3, \( \langle c, u_1, v_1, v_2, x, y_1, y_2 \rangle \) is isomorphic to \( F_{18} \), which is a contradiction. If a vertex \( x \in C \setminus \{c\} \) has two distinct neighbors \( b_1, b_2 \in B \), then \( \langle b_1, b_2, c, u_1, v_1, v_2, x \rangle \) is isomorphic to \( F_{20} \) by noting that
\( E(B) = \emptyset \), a contradiction. So, \( E(B, C \setminus \{c\}) \) is a matching. Now, in view of Facts 4, 6, and 7, it is easily seen that \( G \in G_1 \).

**Case 1.2.** Assume that \(|Q| = 2\) and there exists no \( x \in C \) with \( N(x) = B \).

The existence of \( c \) forces that \(|B| \geq 3\). Fix the distinct vertices \( b_1, b_2, b_3 \in B \) with \( b_1, b_2 \in N_B(c) \) and \( b_3 \notin N_B(c) \). If there exists \( x \in B \setminus \{b_1, b_2, b_3\} \), then \( \langle b_1, b_2, b_3, c, u_1, v_1, v_2, x \rangle \) is isomorphic to \( F_{24} \) or \( F_{25} \), a contradiction. Hence, \( B = \{b_1, b_2, b_3\} \) and \( N_B(c) = \{b_1, b_2\} \). In order to determine \( \langle B, C_B\rangle \), suppose, toward a contradiction, that a vertex \( x \in C \) has exactly one neighbor in \( B \), say \( y \). If \( y \in \{b_1, b_2\} \), then \( \langle b_3, c, u_1, v_1, v_2, x, y \rangle \) is isomorphic to \( F_{19} \), and if \( y = b_3 \), then \( \langle b_1, b_2, b_3, c, v_1, x \rangle \) is isomorphic to \( F_{11} \). This contradiction shows that each vertex in \( C_B \) has exactly two neighbors in \( B \). Moreover, if a vertex \( x \in B \) has three distinct neighbors \( c_1, c_2, c_3 \in C \), then \( \langle c_1, c_2, c_3, u_1, v_1, v_2, x \rangle \) is isomorphic to \( F_{18} \), a contradiction. Thus, each vertex in \( B \) has at most two neighbors in \( C \), which in turn implies that \(|C_B| \leq 3\). Therefore, since \( G \) has no induced subgraph isomorphic to \( F_{20} \), it is straightforwardly checked that \( \langle B, C_B \rangle \) is one of \( K_1 \cup P_3, P_5 \), or \( C_6 \) depending on whether \(|C_B| \) is equal to 1, 2, or 3, respectively.

**Case 1.2.1.** Assume that \( \langle B, C_B \rangle = K_1 \cup P_3 \).

By considering the clique \( \{b_3, v_1, v_2\} \) and by using (??), it is immediately evident that we must have \( N_A(b_3) = A \). Moreover, if \( b_1 \) is not adjacent to a vertex \( x \in A \), then \( \langle b_1, b_3, c, u_1, v_1, x \rangle \) is isomorphic to \( F_{11} \), a contradiction. This yields that \( N_A(b_1) = A \), and similarly, \( N_A(b_2) = A \). Now, it is easily seen from Facts 6 and 7 that \( G \) is obtained by deleting two vertices in \( S_3 \) of a member of the family \( G_2 \).

**Case 1.2.2.** Assume that \( \langle B, C_B \rangle = P_5 \).

Let \( N_C(b_1) = \{c_1, c_2\} \) and \( N_C(b_2) = \{c_1\} \). If \( b_1 \) is not adjacent to a vertex \( x \in A \), then \( \langle b_1, c_1, c_2, u_1, v_1, x \rangle \) is isomorphic to \( F_8 \), a contradiction. This means that \( N_A(b_1) = A \). It follows from Fact 5 that \(|N_A(b_2)| \geq |A| - 1\) and \(|N_A(b_3)| \geq |A| - 1\). If there exists \( x \in N_A(b_2) \setminus N_A(b_3) \), then \( \langle b_2, b_3, c_2, u_1, v_1, x \rangle \) is isomorphic to \( F_{11} \), a contradiction. This implies that \( A \setminus N_A(b_2) = A \setminus N_A(b_3) \). Now, it is easily checked from Facts 6 and 7 that if \( A = N_A(b_2) \), then \( G \) is obtained by deleting one vertex in \( S_3 \) of a member of the family \( G_2 \), and moreover, if \( A \neq N_A(b_2) \), then \( G \in G_3 \) by applying Fact 4.

**Case 1.2.3.** Assume that \( \langle B, C_B \rangle = C_6 \).

If two vertices \( x \in A \) and \( y \in B \) are not adjacent to each other, then \( \langle c_1, c_2, u_1, v_1, x, y \rangle \) with \( N_C(y) = \{c_1, c_2\} \) is isomorphic to \( F_8 \), a contradiction. Therefore, \( \langle A, B \rangle \) is a complete bipartite graph. Now, we find from Facts 6 and 7 that \( G \in G_2 \).

**Case 1.3.** Assume that \(|Q| \geq 3\).

If there is \( x \in B \setminus N(c) \), then \( \langle b_1, b_2, c, u_1, v_1, v_2, v_3, x \rangle \), for some \( b_1, b_2 \in N_B(c) \), is isomorphic to \( F_{26} \), a contradiction. Hence, \( N_B(c) = B \). If two vertices \( x \in A \) and \( y \in A \cup C \) are adjacent to each other, then \( \langle u_1, v_1, v_2, v_3, x, y \rangle \) is isomorphic to \( F_{16} \) or \( F_{17} \), a contradiction. This means that \( E(\langle A \rangle) = \emptyset \) and \( C_A = \emptyset \). If there exists \( x \in C_B \setminus \{c\} \), then \( \langle c, u_1, v_1, v_2, v_3, x, y \rangle \), for some \( y \in N_B(x) \), is isomorphic to \( F_{21} \), a contradiction. Thus, \( C_B = \{c\} \). Further, if a vertex \( x \in A \) is not adjacent to two distinct vertices \( b_1, b_2 \in B \), then \( E(\langle B \rangle) = \emptyset \) yields that \( \langle b_1, b_2, c, u_1, v_1, x \rangle \see
is isomorphic to $F_{11}$, a contradiction. This and Fact 5 together establish that $\langle A, B \rangle$ is the graph obtained from a complete bipartite graph after removing the edges of a matching. Now, Facts 5 and 7 show that $G \in \mathcal{G}_4$. Notice that, if we consider $G$ as Figure ??, then we will have at least one dotted thin line since $\{c, u_1\} \notin \mathcal{E}(G)$.

In the remaining cases, we may assume that every vertex in $C$ has degree 1.

**Case 2.** Assume that $A \cup B$ contains a vertex with at least two neighbors in $C$.

Without loss of generality, assume that there exists $b \in B$ with $|\mathcal{N}_C(b)| \geq 2$. Fix $c_1, c_2 \in \mathcal{N}_C(b)$. If $|P| \geq 2$, then $\langle b, c_1, c_2, u_1, u_2, v_1 \rangle$ is isomorphic to $F_{10}$, a contradiction. So, $|P| = 1$ and hence $|Q| \geq 2$. Moreover, the maximality of $\Omega$ forces that $\mathcal{E}(\langle B \rangle) = \emptyset$. If there is $x \in B \setminus \{b\}$, then $\langle b, c_1, c_2, u_1, v_1, v_2, x \rangle$ is isomorphic to $F_{16}$, since $c_1$ and $c_2$ have the degree 1. This contradiction shows that $B = \{b\}$. If there is $x \in Q \setminus \{v_1, v_2\}$, then $\langle b, c_1, c_2, u_1, v_1, v_2, x \rangle$ is isomorphic to $F_{21}$, a contradiction. Therefore, $Q = \{v_1, v_2\}$. If there is $x \in \mathcal{N}_C(b) \setminus \{c_1, c_2\}$, then $\langle b, c_1, c_2, u_1, v_1, v_2, x \rangle$ is isomorphic to $F_{18}$, a contradiction. This means that $\mathcal{N}_C(b) = C_B = \{c_1, c_2\}$. If $b$ is not adjacent to a vertex $x \in A$, then $\langle b, c_1, c_2, u_1, v_1, x \rangle$ is isomorphic to $F_8$, a contradiction. This shows that $\mathcal{N}_A(b) = A$. Now, by applying Facts 6 and 7, one can easily deduce that $G$ is the graph obtained by deleting two vertices in $S'_3$ and their common neighbor in $S_3$ of a member of the family $\mathcal{G}_2$.

**Case 3.** Assume that $\mathcal{E}(\langle A \cup B, C \rangle)$ is a matching.

We distinguish the following cases.

**Case 3.1.** Assume that $|P| = 1$.

Since $|\Omega| \geq 3$, we have $|Q| \geq 2$. Also, the maximality of $\Omega$ forces that $\mathcal{E}(\langle B \rangle) = \emptyset$. We claim that, if $A$ and $B$ are not empty, then there exists a subset $A' \subseteq A$ with $|A'| \geq |A| - 1$ such that $\langle A', B \rangle$ is a complete bipartite graph and $\mathcal{E}(\langle A \setminus A', B \rangle) = \emptyset$. To see this, consider a vertex $a \in A$ with $\mathcal{N}_B(a) \notin \emptyset, B$. For a vertex $b \in B \setminus \mathcal{N}(a)$, using (??) and by considering the clique $Q \cup \{b\}$, we conclude that a vertex $c \in \mathcal{N}_C(b)$ must exist. But, the subgraph $\langle a, b, b', c, u_1, v_1 \rangle$, for some $b' \in \mathcal{N}_B(a)$, is isomorphic to $F_{11}$, a contradiction. Therefore, for each vertex $a \in A$, either $\mathcal{N}_B(a) = \emptyset$ or $\mathcal{N}_B(a) = B$. If $B \neq \emptyset$ and the first case occurs for two distinct vertices in $A$, then we find a vertex $b \in B$ with $|\mathcal{N}_A(b)| \leq |A| - 2$. This contradicts (??), by considering the clique $Q \cup \{b\}$. So, we proved the claim. Now, we want to show that if $B \neq \emptyset$ and $A \neq A'$, then the unique vertex in $A \setminus A'$ is of degree 1. To see this, consider the vertices $a \in A \setminus A'$, $b \in B$, and $x \in \mathcal{N}_{A \cup C}(a)$. If $x \in A$, then, by the above claim, $\langle b, x \rangle \in \mathcal{E}(G)$ and so $\langle a, b, u_1, v_1, x \rangle$ is isomorphic to $F_5$, a contradiction. If $x \in C$, then $\langle a, b, u_1, v_1, v_2, x \rangle$ is isomorphic to $F_{14}$, which is again a contradiction. This proves what we want. Hence, Facts 6 and 7 imply that $G \in \mathcal{G}_1$ if $|Q| = 2$. So, assume that $|Q| \geq 3$. If a vertex $a \in A$ has a neighbor $x \in A \cup C$, then $\langle a, u_1, v_1, v_2, v_3, x \rangle$ is isomorphic to $F_{16}$ or $F_{17}$, a contradiction. Therefore, in this case, $G \in \mathcal{G}_5$ by using Fact 7.

**Case 3.2.** Assume that $|P| = |Q| = 2$.

Working towards a contradiction, suppose that a vertex $a \in A$ is not adjacent to two distinct vertices $b_1, b_2 \in B$. Then, $\langle a, b_1, b_2, u_1, u_2, v_1 \rangle$ is isomorphic to $F_{15}$ if $\{b_1, b_2\} \notin \mathcal{E}(G)$, and moreover, $\langle a, b_1, b_2, u_1, u_2, v_1, v_2 \rangle$ is isomorphic to $F_{22}$ if $\{b_1, b_2\} \notin \mathcal{E}(G)$. This contradiction shows that $|\mathcal{N}_B(a)| \geq |B| - 1$ for any $a \in A$, and similarly, one can prove that $|\mathcal{N}_A(b)| \geq |A| - 1$. 19
for any \( b \in B \). From Fact 4, each vertex in \( A \) with a neighbor in \( A \cup C \) is adjacent to all vertices in \( B \) as well as each vertex in \( B \) with a neighbor in \( B \cup C \) is adjacent to all vertices in \( A \). Now, Facts 6 and 7 imply that \( G \in \mathcal{G}_1 \).

**Case 3.3.** Assume, without loss of generality, that \( |P| \geq 2 \) and \( |Q| \geq 3 \).

If there are two non-adjacent vertices \( a \in A \) and \( b \in B \), then \( \langle a, b, u_1, v_1, v_2, v_3 \rangle \) is isomorphic to \( F_{23} \), a contradiction. Thus, \( \langle A, B \rangle \) is a complete bipartite graph. If there are two adjacent vertices \( a \in A \) and \( x \in A \cup C \), then \( \langle a, u_1, v_1, v_2, v_3, x \rangle \) is isomorphic to \( F_{16} \) or \( F_{17} \), a contradiction. Hence, no vertex in \( A \) has neighbor in \( A \cup C \). Similarly, we can get that no vertex in \( B \) has neighbor in \( B \cup C \) if \( |P| \geq 3 \). Therefore, we find from Fact 7 that \( G \in \mathcal{G}_5 \) if \( |P| = 2 \) and \( G \in \mathcal{G}_6 \) if \( |P| \geq 3 \).

The proof of the theorem is completed here.

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**References**


